Journal of Approximation Theory **99**, 344–368 (1999)
Article ID jath.1999.3337, available online at http://www.idealibrary.com on IDEAL®

Peano Kernel Error Analysis for Quadratic Nodal Spline Interpolation

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Communicated by Günther Nürnberger

Received March 31, 1998; accepted in revised form November 5, 1998

Using a Peano kernel technique, Jackson-type estimates with respect to the maximum norm are derived for the quadratic nodal spline interpolation error. The explicitly calculated error constants are shown to grow linearly with respect to the local mesh ratio parameter, and are, at least for the important special case of a uniform spline knot sequence, significantly smaller than those previously calculated by different methods. © 1999 Academic Press

1. INTRODUCTION

In a sequence of five papers [5–9] de Villiers and Rohwer introduced and analyzed an arbitrary order nodal spline approximation operator with the favourable properties of locality, interpolation at a subsequence of the spline knots, as well as optimal order polynomial reproduction. Also, in [8], it was shown that, in the context of quadrature rules, nodal spline interpolation yields an interpolant for the Gregory rule, which is an important example of a trapezoidal rule with endpoint corrections. Other recent papers in which the approximation properties and applications in quadrature of nodal spline interpolation were further explored, include those by Rabinowitz [10, 11], Demichelis [4], and Dagnino *et al.* [2]. Fundamental existence and uniqueness theorems for spline interpolation by means of additional knots, including the nodal spline case, were proved by Dahmen *et al.* [3], and an explicit construction procedure for some of these spline interpolation operators was introduced by Chui and de Villiers in [1].

In this paper we consider specifically the *quadratic* nodal spline interpolation error, and proceed to show how a Peano kernel technique can be

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0021-9045/99 $30.00

used to establish Jackson-type estimates for the maximum norm error in which the error constants are, at least for the case of uniformly distributed spline knots, significantly smaller than those previously calculated by different methods in [7, 9]. First, we introduce some notation, followed by a summary of those results from [5–9] which will be needed in our work.

Suppose that $[a, b]$ is a finite interval on the real line $\mathbb{R}$ and, for a given integer $n \geqslant 2$, let the finite sequences $\{x_i = x_{n,i} : i = 0, 1, ..., 2n\}$ and $\{\xi_i = \xi_{n,i} : i = 0, 1, ..., n\}$ be such that

$$\Delta_n : a = x_0 < x_1 < \cdots < x_{2n} = b, \tag{1.1}$$

and

$$\begin{aligned} &\Pi_n : a = \xi_0 < \xi_1 < \cdots < \xi_n = b, \\ &\text{with} \quad \xi_i = x_{2i}, \qquad i = 0, 1, ..., n, \end{aligned} \tag{1.2}$$

denote partitions of $[a, b]$. The points of Δ_n and $\Delta_n \backslash \Pi_n$ will be called, respectively, the primary and secondary knots corresponding to the partition Δ_n. We use the symbol $\mathbb{P}^2$ for the set of polynomials of degree $\leqslant 2$, whereas $S(\Delta_n)$ will denote the set of polynomial splines of order 3 (i.e., degree $\leqslant 2$) with simple knots at the points $\{x_i : i = 1, 2, ..., 2n-1\}$, so that $S(\Delta_n) \subset C^1[a, b]$.

As introduced in [5, 6], the quadratic nodal spline approximation operator $V = V_n : C[a, b] \to S(\Delta_n)$ is defined by

$$(Vf)(x) = \sum_{i=p_j}^{q_j} f(\xi_i)\, v_i(x), \qquad x \in [\xi_j, \xi_{j+1}], \qquad j = 0, 1, ..., n-1, \tag{1.3}$$

with

$$p_j := \max\{0, j-1\}, \qquad q_j := \min\{n, j+2\}; \tag{1.4}$$

where the quadratic spline functions $\{v_i = v_{n,i} : i = 0, 1, ..., n\} \subset S(\Delta_n)$ satisfy

$$v_i(x) = \begin{cases} \displaystyle\prod_{i \neq k = 0}^{2} \frac{x - \xi_k}{\xi_i - \xi_k}, & x \in [a, \xi_1], \qquad i = 0, 1, 2, \\ s_i(x), \qquad x \in [\xi_1, \xi_{n-1}], & i = 0, 1, ..., n, \quad (\text{if } n \geqslant 3), \\ \displaystyle\prod_{n-i \neq k = 0}^{2} \frac{x - \xi_{n-k}}{\xi_i - \xi_{n-k}}, & x \in [\xi_{n-1}, b], \qquad i = n-2, n-1, n; \end{cases} \tag{1.5}$$

and where the quadratic nodal splines $\{s_i = s_{n,i} : i = 0, 1, ..., n\} \subset S(\Delta_n)$ can be calculated on $[\xi_1, \xi_{n-1}]$ from the formulas

$$s_i(x) = \begin{cases} s(x;\, \xi_{i-1}, \xi_i, x_{2i+1}, \xi_{i+1}, x_{2i+3}, \xi_{i+2}), \\ \qquad x \in [\xi_i, \xi_{n-1}], \\ s(-x;\, -\xi_{i+1}, -\xi_i, -x_{2i-1}, -\xi_{i-1}, -x_{2i-3}, -\xi_{i-2}), \\ \qquad x \in [\xi_1, \xi_i], \end{cases} \tag{1.6}$$

with

$$s(x;\, \xi_{i-1}, \xi_i, x_{2i+1}, \xi_{i+1}, x_{2i+3}, \xi_{i+2})$$
$$= \begin{cases} 1 + (x-\xi_i)[A_i - C_i(x-\xi_i)], & x \in [\xi_i, x_{2i+1}], \\ (\xi_{i+1}-x)[B_i + D_i(\xi_{i+1}-x)], & x \in [x_{2i+1}, \xi_{i+1}], \\ (x-\xi_{i+1})[-B_i + E_i(x-\xi_{i+1})], & x \in [\xi_{i+1}, x_{2i+3}], \\ -F_i(\xi_{i+2}-x)^2, & x \in [x_{2i+3}, \xi_{i+2}], \\ 0, & x \in [\xi_{i+2}, \xi_{n-1}], \end{cases} \tag{1.7}$$

and

$$A_i = \frac{(\xi_{i+1}-\xi_i)-(\xi_i-\xi_{i-1})}{(\xi_{i+1}-\xi_i)(\xi_i-\xi_{i-1})},$$

$$B_i = \frac{\xi_{i+2}-\xi_{i+1}}{(\xi_{i+2}-\xi_i)(\xi_{i+1}-\xi_i)},$$

$$C_i = \frac{\left[\begin{array}{r}(\xi_{i+2}-\xi_i)(x_{2i+1}-\xi_i)+(\xi_{i+1}-\xi_i)(\xi_{i+2}-\xi_i) \\ +(\xi_i-\xi_{i-1})(\xi_{i+1}-x_{2i+1})\end{array}\right]}{2(\xi_{i+2}-\xi_i)(\xi_{i+1}-\xi_i)(\xi_i-\xi_{i-1})(x_{2i+1}-\xi_i)},$$

$$D_i = \frac{\left[\begin{array}{r}(\xi_{i+2}-\xi_i)(x_{2i+1}-\xi_i)+(\xi_{i+1}-\xi_i)(\xi_i-\xi_{i-1}) \\ +(\xi_i-\xi_{i-1})(\xi_{i+1}-x_{2i+1})\end{array}\right]}{2(\xi_{i+2}-\xi_i)(\xi_{i+1}-\xi_i)(\xi_i-\xi_{i-1})(\xi_{i+1}-x_{2i+1})}, \tag{1.8}$$

$$E_i = \frac{(x_{2i+3}-\xi_{i+1})+(\xi_{i+2}-\xi_{i+1})}{2(\xi_{i+2}-\xi_i)(\xi_{i+1}-\xi_i)(x_{2i+3}-\xi_{i+1})},$$

$$F_i = \frac{x_{2i+3}-\xi_{i+1}}{2(\xi_{i+2}-\xi_i)(\xi_{i+1}-\xi_i)(\xi_{i+2}-x_{2i+3})}.$$

Note that the (undefined) symbols $\xi_{-2}, \xi_{-1}, \xi_{n+1}, \xi_{n+2}$ appearing in (1.6), (1.7), (1.8) for $i = 0, 1$ and $i = n-1, n$, are *not* needed in the calculation of s_i on $[\xi_1, \xi_{n-1}]$.

Also, observe from (1.6), (1.7) that the nodal property

$$s_i(\xi_i)=\delta_{i,j}, \qquad i=0, 1, ..., n, \qquad j=1, 2, ..., n-1, \tag{1.9}$$

with $\delta_{i,j}$ denoting the Kronecker delta, is satisfied.

It can be shown that V satisfies the following three properties:

[A] V is a *local* approximation operator in the sense that, for a given $x \in [a, b]$, the value of $(Vf)(x)$ depends only on at most four neighbouring values $f(\xi_i)$;

[B] V possesses the property of *interpolation at the primary knots* Π_n,

$$(Vf)(\xi_i)=f(\xi_i), \qquad i=0, 1, ..., n, \qquad f \in C[a, b]; \tag{1.10}$$

[C] V possesses the property of *optimal degree polynomial reproduction*,

$$Vp=p, \qquad p \in \mathbb{P}^2. \tag{1.11}$$

Next, in order to analyze the maximum norm interpolation error

$$\|f-Vf\|_\infty := \max_{a \leqslant x \leqslant b} |f(x)-(Vf)(x)|, \qquad f \in C[a, b],$$

we introduce the following primary mesh parameters corresponding to the partition Π_n of $[a, b]$:

(i) the *primary mesh norm* $H=H_n$ is defined by

$$H := \max\{\xi_{i+1}-\xi_i : i=0, 1, ..., n\}; \tag{1.12}$$

(ii) the *local primary mesh ratio* $R=R_n$ is defined by

$$R := \max\left\{\frac{\xi_{i+1}-\xi_i}{\xi_{j+1}-\xi_j} : 0 \leqslant i, j \leqslant n-1, \ |i-j|=1\right\}. \tag{1.13}$$

Note from (1.13) that $R \geqslant 1$, with $R=1$ if and only if Π_n is the uniform primary partition

$$\xi_i=a+iH, \qquad i=0, 1, ..., n, \qquad H=\frac{b-a}{n}. \tag{1.14}$$

In [7, 9], de Villiers and Rohwer employed a method based on estimating the Lebesgue constant $\|V\|_\infty$ to establish, for $f \in C^r[a, b]$, $r=0, 1, 2, 3$, Jackson-type bounds on the interpolation error $\|f-Vf\|_\infty$, in which the explicit dependence on the local primary mesh ratio parameter R was explicitly calculated.

The principle aim of this paper is to demonstrate how an alternative error analysis method based instead on a Peano kernel technique can be employed, as in analogous previous work in the context of Lagrange polynomial interpolation by Stroud [12, pp. 69–81], to obtain, for $f \in C^r[a, b]$, $r = 1, 2, 3$, error constants in the Jackson-type bounds for $\|f - Vf\|_\infty$ which are, in particular for the uniform primary knot sequence (1.14), significantly smaller than the corresponding error constants obtained in [7, 9].

We shall rely on a Peano kernel result for interpolation error analysis (see also [12, pp. 69–70]) which, in the context of quadratic nodal spline interpolation, where in particular (1.3) and (1.11) hold, and in the usual notation

$$(x - t)_+^k := \begin{cases} (x - t)^k, & x \geqslant t \\ 0, & x < t, \end{cases} \tag{1.15}$$

for the truncated power function, is expressible in the form

$$|(f - Vf)(x)| \leqslant \left[\int_a^b |K_r(x, \theta)| \, d\theta \right] \|f^{(r)}\|_\infty,$$
$$a \leqslant x \leqslant b, \qquad f \in C^r[a, b], \qquad r = 1, 2, 3, \tag{1.16}$$

where

$$K_r(x, \theta) = \frac{1}{(r-1)!} \left[(x - \theta)_+^{r-1} - \sum_{i = p_j}^{q_j} (\xi_i - \theta)_+^{r-1} \, v_i(x) \right],$$
$$x \in [\xi_j, \xi_{j+1}], \; j = 0, 1, ..., n-1. \tag{1.17}$$

We now proceed, according to the steps carried out in Sections 2, 3 and 4 below, to collect the necessary results by virtue of which the desired Jackson-type estimates for quadratic nodal spline interpolation, as given in our main result in Theorem 5.1, will follow.

2. INTERIOR ESTIMATES

Our first step is to estimate the function $\int_a^b |K_r(x, \theta)| \, d\theta$, as appearing in (1.16), for $x \in [\xi_1, \xi_{n-1}]$, i.e., for x bounded away from the endpoints a and b.

In the proof below we shall exploit the fact, as can be seen from (1.3), (1.4), (1.5), together with (1.11), that the quadratic nodal splines $\{s_i : i=0, 1, ..., n\}$ satisfy the moment property

$$\sum_{i=j-1}^{j+2} \xi_i^k s_i(x) = x^k, \qquad x \in [\xi_j, \xi_{j+1}], \qquad j=1, 2, ..., n-2, \qquad k=0, 1, 2; \tag{2.1}$$

and in particular, as can be seen by setting $k=0$ in (2.1), the partition of unity property

$$\sum_{i=j-1}^{j+2} s_i(x) = 1, \qquad x \in [\xi_j, \xi_{j+1}], \qquad j=1, 2, ..., n-2. \tag{2.2}$$

LEMMA 2.1. *Let* $r \in \{1, 2, 3\}$ *and* $j \in \{1, 2, ..., n-2\}$, *and suppose* $x \in [\xi_j, \xi_{j+1}]$. *Then, with* $K_r(x, \theta)$ *given by* (1.17), *we have*

$$\int_a^b |K_r(x, \theta)| \, d\theta$$
$$\leqslant \frac{1}{r!} \left[-e_r(x) + 2 \begin{cases} s_j(x)(x-\xi_j)^r + s_{j+2}(x)(x-\xi_{j+2})^r, & r=1, 3, \\ s_j(x)(x-\xi_j)^r + s_{j+1}(x)(x-\xi_{j+1})^r, & r=2, \end{cases} \right] \tag{2.3}$$

or, alternatively,

$$\int_a^b |K_r(x, \theta)| \, d\theta$$
$$\leqslant \frac{1}{r!} \left[e_r(x) - 2 \begin{cases} s_{j-1}(x)(x-\xi_{j-1})^r + s_{j+1}(x)(x-\xi_{j+1})^r, & r=1, 3, \\ s_{j-1}(x)(x-\xi_{j-1})^r + s_{j+2}(x)(x-\xi_{j+2})^r, & r=2, \end{cases} \right] \tag{2.4}$$

where

$$\left.\begin{aligned} e_r(x) &= \begin{cases} 0 & r=1, 2, \\ (f-Vf)(x), & r=3, \end{cases} \\ \textit{with} \qquad f(x) &= x^3, \end{aligned}\right\} \quad a \leqslant x \leqslant b. \tag{2.5}$$

Proof. For a given $x \in [\xi_j, \xi_{j+1}]$, where $j \in \{1, 2, ..., n-2\}$, it can be deduced from (1.17), (1.5), (1.15), and (2.1), that the estimate

$$\int_a^b |K_r(x, \theta)| \, d\theta \leqslant \frac{1}{r!} \sum_{i=j-1}^{j+2} |s_i(x)| \, |x-\xi_i|^r \tag{2.6}$$

holds.

Next, we fix $r \in \{1, 3\}$ and observe from the sign properties of s_i, as implied by (1.6), (1.7), (1.8), that

$$\sum_{i=j-1}^{j+2} |s_i(x)|\, |x-\xi_i|^r = \sum_{i=j-1}^{j+2} (-1)^{i-j} s_i(x)(x-\xi_i)^r$$
$$= \sum_{\ell=0}^{r} (-1)^{\ell} \binom{r}{\ell} x^{r-\ell} \sum_{i=j-1}^{j+2} (-1)^{i-j} s_i(x)\, \xi_i^{\ell}. \tag{2.7}$$

But, for $\ell \in \{0, 1, 2\}$,

$$\sum_{i=j-1}^{j+2} (-1)^{i-j} s_i(x)\, \xi_i^{\ell} = -\sum_{i=j-1}^{j+2} s_i(x)\, \xi_i^{\ell} + \sum_{i=j-1}^{j+2} [1+(-1)^{i-j}]\, s_i(x)\, \xi_i^{\ell}$$
$$= -x^{\ell} + 2s_j(x)\, \xi_j^{\ell} + 2s_{j+2}(x)\, \xi_{j+2}^{\ell}, \tag{2.8}$$

by virtue of (2.1). Now substitute (2.8) into (2.7) to obtain

$$\sum_{i=j-1}^{j+2} |s_i(x)|\, |x-\xi_i|^r$$
$$= \begin{cases} \displaystyle\sum_{\ell=0}^{1} (-1)^{\ell} \binom{1}{\ell} x^{1-\ell}\, g_{\ell}(x)[-x^{\ell} + 2s_j(x)\xi_j^{l} + 2s_{j+2}(x)\xi_{j+2}^{l}], & r=1, \\ \displaystyle\sum_{\ell=0}^{2} (-1)^{\ell} \binom{3}{\ell} x^{3-\ell}[-x^{\ell} + 2s_j(x)\xi_j^{\ell} + 2s_{j+2}(x)\xi_{j+2}^{\ell}] & \\ \qquad \displaystyle -\sum_{i=j-1}^{j+2} (-1)^{i-j}\, \xi_i^3 s_i(x), & r=3. \end{cases} \tag{2.9}$$

Thus, for $r=1$, (2.3) follows immediately from (2.6), (2.5) and the first line of (2.9); whereas, for $r=3$, the desired result in (2.3) follows by using the second line of (2.9) to obtain

$$\sum_{i=j-1}^{j+2} |x-\xi_i|^3\, |s_i(x)|$$
$$= -x^3 + 2s_j(x)(x-\xi_j)^3 + 2s_{j+2}(x)(x-\xi_{j+2})^3 + \sum_{i=j-1}^{j+2} s_i(x)\, \xi_i^3,$$

and then recalling (2.6) and (2.5).

The estimates (2.4) are proved analogously. ∎

Next, we establish bounds for the terms $s_k(x)(x-\xi_k)^r$, $k=j-1, ..., j+2$, for $x \in [\xi_1, \xi_{n-1}]$, as appearing in (2.3) and (2.4), in terms of the mesh parameters H and R.

It will be convenient to introduce the following notation. For $j, k \in \{1, ..., n-2\}$ and $r \in \{1, 2, 3\}$, we define the constants

$$\sigma_{j,k,r} := \max_{\xi_j \leqslant x \leqslant x_{2j+1}} |s_k(x)(x-\xi_k)^r|, \tag{2.10}$$

and

$$\tau_{j,k,r} := \max_{x_{2j+1} \leqslant x \leqslant \xi_{j+1}} |s_k(x)(x-\xi_k)^r|. \tag{2.11}$$

The following estimates can then be proved.

LEMMA 2.2. *Suppose $j \in \{1, 2, ..., n-2\}$, and $r \in \{1, 2, 3\}$. Then the constants $\sigma_{j,k,r}$ and $\tau_{j,k,r}$ in (2.10) and (2.11) are bounded by*

$$\left.\begin{matrix}\sigma_{j,j,r}\\ \tau_{j,j-1,r}\end{matrix}\right\} \leqslant C_{r,1} H^r, \qquad \textit{and} \qquad \left.\begin{matrix}\sigma_{j,j+2,r}\\ \tau_{j,j+1,r}\end{matrix}\right\} \leqslant C_{r,2} H^r, \qquad r = 1, 3, \tag{2.12}$$

$$\left.\begin{matrix}\sigma_{j,j-1,2}\\ \tau_{j,j+2,2}\end{matrix}\right\} \leqslant C_{2,1} H^2, \qquad \textit{and} \qquad \left.\begin{matrix}\sigma_{j,j+2,2}\\ \tau_{j,j-1,2}\end{matrix}\right\} \leqslant C_{2,2} H^2, \tag{2.13}$$

with the positive numbers $C_{r,i} = C_{r,i}(R)$ given by the formulas

$$C_{1,1} = \frac{1}{27}\left[\frac{(R-1)(2R+1)(R+2) + 2(R^2+R+1)^{3/2}}{R^2}\right], \tag{2.14}$$

$$C_{1,2} = \frac{(3+\sqrt{3})}{54}\left[\frac{R^2-2R-2+(R+2)\sqrt{R^2+R+1}}{R+1}\right], \tag{2.15}$$

$$C_{2,1} = \frac{(3\sqrt{17}-5)}{2048}\,\frac{[3R+6+\sqrt{9R^2+4R+4}]^2}{R+1}, \tag{2.16}$$

$$C_{2,2} = \frac{(3\sqrt{17}-5)}{2048}\left[\frac{9R^2+20R+20+3(R+2)\sqrt{9R^2+4R+4}}{R+1}\right], \tag{2.17}$$

$$C_{3,1} = \frac{1}{3125}\,\frac{\left[\begin{matrix}[2R-2+\sqrt{4R^2+7R+4}]^3 \\ \times[2R^2+6R+2+(R-1)\sqrt{4R^2+7R+4}]\end{matrix}\right]}{R^4}, \tag{2.18}$$

$$C_{3,2} = \frac{(3+8\sqrt{6})}{3125}\left[\frac{8R^2+17R+17+4(R+2)\sqrt{4R^2+4R+1}}{R+1}\right]. \tag{2.19}$$

Proof. We shall omit many details of the (long and mostly rather technical) proof for the bounds on $\sigma_{j,k,r}$, due to the fact that routine calculus procedures are used throughout; also, the proofs for $\tau_{j,k,r}$ (i.e., the bounds on $[x_{2j+1}, \xi_{j+1}]$) are similar, and therefore omitted altogether.

Hence, throughout the proof, we fix $j \in \{1, ..., n-2\}$ and suppose $x \in [\xi_j, x_{2j+1}]$. Then, introducing the notation

$$\begin{aligned} y &:= x - \xi_j, & z &:= x - \xi_{j+1}, \\ \alpha &:= \xi_j - \xi_{j-1}, & \beta &:= \xi_{j+1} - \xi_j, \\ \gamma &:= \xi_{j+2} - \xi_{j+1}, & h &:= x_{2j+1} - \xi_j, \end{aligned} \tag{2.20}$$

the formulas (1.6), (1.7), (1.8) can be used to obtain

$$\sigma_{j,k,r} = \max_{0 \leqslant y \leqslant h} p_{k,r}(y), \tag{2.21}$$

with

$$\begin{aligned} p(y) &:= p_{j,1}(y) = y(1 + Ay - Cy^2), \\ q(y) &:= p_{j+2,1}(y) = Fy^2(\beta + \gamma - y), \\ P(y) &:= p_{j-1,2}(y) = y(B - Ey)(\alpha + y)^2, \\ Q(y) &:= p_{j+2,2}(y) = (\beta + \gamma - y)\, q(y) = Fy^2(\beta + \gamma - y)^2, \\ M(y) &:= p_{j,3}(y) = y^2 p(y) = y^3(1 + Ay - Cy^2), \\ N(y) &:= p_{j+2,3}(y) = (\beta + \gamma - y)^2\, q(y) = Fy^2(\beta + \gamma - y)^3, \end{aligned} \tag{2.22}$$

and where

$$\begin{aligned} A &= \frac{\beta - \alpha}{\alpha\beta}, \qquad B = \frac{\beta}{\alpha(\alpha + \beta)}, \qquad C = \frac{\beta(\alpha + \beta + \gamma) + h(\beta + \gamma - \alpha)}{2\alpha\beta(\beta + \gamma)\, h}, \\ E &= \frac{\beta + h}{2\alpha(\alpha + \beta)\, h}, \qquad F = \frac{1}{2\gamma(\beta + \gamma)} \left(\frac{\beta}{h} - 1\right). \end{aligned} \tag{2.23}$$

We now systematically proceed to first maximize the functions $p(y)$, $q(y)$, ..., $N(y)$, as given by (2.22), and then bound he resulting maxima in terms of the primary mesh parameters H and R.

(a) *The Case* $r = 1$.

(i) From (2.2), (2.22) we find that

$$\begin{aligned} \sigma_{j,j,1} &= p\left(\frac{A + \sqrt{A^2 + 3C}}{3C}\right) \\ &= \frac{1}{27}\left(\frac{A}{C} + \sqrt{\frac{A^2}{C^2} + \frac{3}{C}}\right)\left(6 + \frac{A^2}{C} + A\sqrt{\frac{A^2}{C^2} + \frac{3}{C}}\right). \end{aligned} \tag{2.24}$$

But, using the definition of C in (2.23), we deduce that, since also $0 < h < \beta$,

$$0 < \frac{1}{C} = \frac{2\alpha\beta(\beta+\gamma)}{\beta(\alpha+\beta+\gamma)/h+\beta+\gamma-\alpha} < \alpha\beta,$$

which, inserted into (2.24) together with the value of A given in (2.23), yields, in terms of the ratio $v := \beta/\alpha$, the bound

$$\sigma_{j,\,j,\,1} < \tfrac{1}{27}\, g(\alpha, \beta)\; G(v), \tag{2.25}$$

with

$$g(\alpha, \beta) = \beta - \alpha + \sqrt{\alpha^2 + \alpha\beta + \beta^2}, \qquad G(v) = 4 + v + \frac{1}{v} + \left(1 - \frac{1}{v}\right)\sqrt{v^2 + v + 1}.$$

Since $(\partial/\partial\alpha)\, g(\alpha, \beta) < 0$ for $\alpha > 0$, $\beta > 0$, and $G'(v) > 0$ for $v > 0$, we get

$$g(\alpha, \beta) \leqslant g\left(\frac{\beta}{R}, \beta\right) \leqslant \left[1 - \frac{1}{R} + \sqrt{\frac{1}{R^2} + \frac{1}{R} + 1}\right] H,$$

and (2.26)

$$G(v) \leqslant G(R) = 4 + R + \frac{1}{R} + \left(1 - \frac{1}{R}\right)\sqrt{R^2 + R + 1}.$$

Now combine (2.25) and (2.26) to deduce the desired bound, as given in (2.12) and (2.14), on $\sigma_{j,\,j,\,1}$.

(ii) Next, to bound $\sigma_{j,\,j+2,\,1}$, we analyze the polynomial $q(y)$ in (2.22) to find, in the notation $\delta := \beta + \gamma$, that

$$\sigma_{j,\,j+2,\,1} = \begin{cases} q(\tfrac{2}{3}\delta), & \text{if } \tfrac{2}{3}\delta \leqslant h < \beta, \\ q(h), & \text{if } 0 < h < \tfrac{2}{3}\delta. \end{cases} \tag{2.27}$$

Suppose first that $\tfrac{2}{3}\delta \leqslant h < \beta$. Then, from (2.22), (2.23), we get

$$q\left(\frac{2}{3}\,\delta\right) = \frac{2}{27}\,\frac{1}{\gamma}\left(\frac{\beta}{h} - 1\right)(\beta+\gamma)^2 \leqslant \frac{1}{27}\, F(\beta, \gamma)\, \beta, \tag{2.28}$$

where

$$\left(\frac{\beta}{\gamma} - 2\right)\left(\frac{\gamma}{\beta} + 1\right) =: F(\beta, \gamma) \leqslant F\left(\beta, \frac{\beta}{R}\right) = (R-2)\left(\frac{1}{R} + 1\right),$$

so that (2.28) gives the bound

$$q\left(\frac{2}{3}\,\delta\right) \leqslant \frac{1}{27}\,(R-2)\left(\frac{1}{R}+1\right) H. \tag{2.29}$$

Next, for $0 < h \leqslant \frac{2}{3}\delta$, we calculate from (2.22), (2.23) that, in the notation $\mu := \beta/\gamma$,

$$\max_{0 < h \leqslant (2/3)\delta} q(h) \leqslant q\left(\frac{2\beta+\gamma-\sqrt{\beta^2+\beta\gamma+\gamma^2}}{3}\right) = \frac{1}{54}\,U(\beta,\gamma)\,W(\mu),$$

with

$$U(\beta,\gamma) = \beta+2\gamma+\sqrt{\beta^2+\beta\gamma+\gamma^2},$$
$$W(\mu) = \frac{\mu^2-2\mu-2+(\mu+2)\sqrt{\mu^2+\mu+1}}{\mu+1}.$$

We can then show that

$$U(\beta,\gamma) \leqslant U(H,H) = (3+\sqrt{3})\,H,$$
$$W(\mu) \leqslant W(R) = \frac{R^2-2R-2+(R+2)\sqrt{R^2+R+1}}{R+1},$$

and it follows that $q(h) \leqslant C_{1,2}H$, with $C_{1,2}$ defined by (2.15). But, since

$$2(R-2)\left(\frac{1}{R}+1\right)$$
$$\leqslant (3+\sqrt{3})\,\frac{[R^2-2R-2+(R+2)\sqrt{R^2+R+1}]}{R+1}, \qquad R \geqslant 1,$$

we conclude, recalling also (2.27) and (2.29), that the inequality $\sigma_{j,\,j+2,\,1} \leqslant C_{1,2}H$ in (2.12) holds.

(b) *The Case* $r=2$.

(i) For $\sigma_{j,\,j-1,\,2}$ we have, from (2.21), (2.22), and with the notation $\lambda := B/E$, that

$$\sigma_{j,\,j-1,\,2} = \begin{cases} P(y_+), & \text{if} \quad y_+ \leqslant h, \\ P(h), & \text{if} \quad y_+ \geqslant h, \end{cases} \tag{2.30}$$

where

$$y_+ = \frac{3\lambda-2\alpha+\sqrt{9\lambda^2+4\lambda\alpha+4\alpha^2}}{8}.$$

Suppose first that $y_+ \geqslant h$, in which case we can show that

$$\max_{0<h<\beta} P(h) = P\left(\frac{3\beta - 2\alpha + \sqrt{9\beta^2 + 4\beta\alpha + 4\alpha^2}}{8}\right) = \frac{1}{2048}\, \hat{U}(\alpha, \beta)\, \hat{W}(v), \tag{2.31}$$

with

$$\begin{aligned} \hat{U}(\alpha, \beta) &= 3\beta^2 - 4\alpha\beta - 4\alpha + (\beta + 2\alpha)\sqrt{9\beta^2 + 4\alpha\beta + 4\alpha^2}, \\ \hat{W}(v) &= \frac{[3v + 6 + \sqrt{9v^2 + 4v + 4}]^2}{v+1}, \end{aligned} \tag{2.32}$$

for which we can deduce that

$$\begin{aligned} \hat{U}(\alpha, \beta) &\leqslant \hat{U}(H, H) = (3\sqrt{17} - 5)\, H^2, \\ \hat{W}(v) &\leqslant \hat{W}(R) = \frac{[3R + 6 + \sqrt{9R^2 + 4R + 4}]^2}{R+1}. \end{aligned} \tag{2.33}$$

Substitution of (2.33) into (2.31) then yields, together with (2.30),

$$\sigma_{j,\, j-1,\, 2} \leqslant \frac{(3\sqrt{17} - 5)}{2048}\, \frac{[3R + 6 + \sqrt{9R^2 + 4R + 4}]^2}{R+1}\, H^2, \qquad \text{if} \quad y_+ \geqslant h. \tag{2.34}$$

For the case $y_+ \leqslant h$, we calculate, using also the fact that $0 < \lambda = 2\beta h/(\beta + h) < \beta$ for $0 < h < \beta$, the bound

$$P(y_+) \leqslant \tfrac{1}{2048}\, \hat{U}(\alpha, \beta)\, \hat{W}(v), \tag{2.35}$$

with $\hat{U}(\alpha, \beta)$ and $\hat{W}(v)$ defined as in (2.32). Hence the desired inequality $\sigma_{j,\, j-1,\, 2} \leqslant C_{2,1} H^2$ in (2.13), with $C_{2,1}$ as defined by (2.16), then immediately follows by noting from (2.35), (2.31), (2.30), that the bound in (2.34) also holds for $y_+ \leqslant h$.

(ii) To bound $\sigma_{j,\, j+2,\, 2}$ we use (2.21), (2.22), (2.23) to deduce that

$$\sigma_{j,\, j+2,\, 2} = \begin{cases} Q\left(\dfrac{\delta}{2}\right), & \dfrac{\delta}{2} \leqslant h < \beta, \\[2ex] Q(h), & 0 < h < \dfrac{\delta}{2}, \end{cases} \tag{2.36}$$

with, as before, $\delta := \beta + \alpha$.

Suppose first that $\delta/2 \leqslant h < \beta$. Then, as in the argument leading from (2.28) to (2.29), we find that

$$Q\left(\frac{\delta}{2}\right) \leqslant \frac{1}{32}\,(R-1)\left(\frac{1}{R}+1\right)^2 H^2. \tag{2.37}$$

Next, for $0 < h \leqslant \delta/2$, we proceed to show from (2.22), (2.23), and exploiting also the similarity here to the maximization procedure leading to (2.31), that

$$Q(h) \leqslant \frac{(3\sqrt{17}-5)}{4096}\,\frac{[3R+6+\sqrt{9R^2+4R+4}]^2}{R+1}\,H^2 \qquad \text{if} \quad 0 < h \leqslant \frac{\delta}{2}. \tag{2.38}$$

The inequality (2.13) for $\sigma_{j,\,j+2,\,2}$ can then be deduced form (2.37), (2.38), and (2.36).

(c) *The Case* $r=3$.

(i) For $\sigma_{j,\,j,\,3}$ we find from (2.21), (2.22), (2.23) that, with $v := \beta/\alpha$ as before, and following a procedure similar to the one leading from (2.24) to (2.25),

$$\sigma_{j,\,j,\,3} = M\left(\frac{2A+\sqrt{4A^2+15C}}{5C}\right) < \frac{1}{3125}\,[\,\hat{g}(\alpha,\beta)]^3\,\hat{G}(v), \tag{2.39}$$

where

$$\hat{g}(\alpha,\beta) = 2\beta - 2\alpha + \sqrt{4\alpha^2 + 7\alpha\beta + 4\beta^2},$$

$$\hat{G}(v) = 6 + 2v + \frac{2}{v} + \left(1 - \frac{1}{v}\right)\sqrt{4v^2 + 7v + 4}.$$

Here we can show that

$$\hat{g}(\alpha,\beta) \leqslant \hat{g}\left(\frac{\beta}{R},\beta\right) \leqslant \left[2 - \frac{2}{R} + \sqrt{\frac{4}{R^2} + \frac{7}{R} + 4}\,\right] H, \tag{2.40}$$

and

$$\hat{G}(v) \leqslant \hat{G}(R) = 6 + 2R + \frac{2}{R} + \left(1 + \frac{1}{R}\right)\sqrt{4R^2 + 7R + 4}, \tag{2.41}$$

so that the desired bound (2.12), (2.18) for $\sigma_{j,\,j,\,3}$ is an immediate consequence of (2.39), (2.40), (2.41).

(ii) Finally, to bound $\sigma_{j,\,j+2,\,3}$, we use (2.21), (2.22) to find that

$$\sigma_{j,\,j+2,\,3}\begin{cases} N\left(\dfrac{2\delta}{5}\right), & \dfrac{2\delta}{5}\leqslant h<\beta, \\ N(h), & 0<h<\dfrac{2\delta}{5}, \end{cases} \tag{2.42}$$

with δ defined as before by $\delta := \beta + \alpha$.

Suppose first that $2\delta/5 \leqslant h < \beta$. Then, similarly to the argument leading form (2.28) to (2.29), we can deduce the bound

$$N\left(\frac{2\delta}{5}\right)\leqslant \frac{27}{3125}\,(3R-2)\left(\frac{1}{R}+1\right)^3 H^3. \tag{2.43}$$

Next, suppose that $0<h\leqslant 2\delta/5$, in which case we find, from (2.22) and (2.23), that

$$\max_{0<h\leqslant 2\delta/5} N(h)\leqslant N\left(\frac{3\beta+\gamma-\sqrt{4\beta^2+\beta\gamma+\gamma^2}}{5}\right)=\frac{1}{6250}\,X(\beta,\gamma)\;Y(\beta,\gamma)\;Z(\mu), \tag{2.44}$$

with

$$\begin{aligned} X(\beta,\gamma)&=2(\beta^2-\beta\gamma-\gamma^2)+(\beta+2\gamma)\sqrt{4\beta^2+\beta\gamma+\gamma^2}, \\ Y(\beta,\gamma)&=2\beta+4\gamma+\sqrt{4\beta^2+\beta\gamma+\gamma^2}, \\ Z(\mu)&=\frac{(2\mu+4+\sqrt{4\mu^2+\mu+1})^2}{\mu+1}, \end{aligned}$$

and where, as before, $\mu := \beta/\gamma$.

Here we can show that

$$\begin{aligned} X(\beta,\gamma)&\leqslant X(H,H)=(3\sqrt{6}-2)\,H^2, \\ Y(\beta,\gamma)&\leqslant Y(H,H)=(\sqrt{6}+6)\,H, \\ Z(\mu)&\leqslant Z(R)=\frac{(2R+4+\sqrt{4R^2+R+1})^2}{R+1}. \end{aligned} \tag{2.45}$$

Substitution of (2.45) into (2.44) then shows that $N(h)$ is bounded above by $C_{3,\,2}H^3$ with $C_{3,\,2}$ as defined by (2.19), and thus, recalling also (2.42) and (2.43), we conclude that the inequality $\sigma_{j,\, j+2,\, 3} \leqslant C_{3,\, 2} H^3$ holds by virtue of the fact that

$$27(3R-2)\left(\frac{1}{R}+1\right)^3 \leqslant (3+8\sqrt{6})\,\frac{(2R+4+\sqrt{4R^2+R+1})^2}{R+1}, \qquad R \geqslant 1. \quad \blacksquare$$

Remarks. (a) For the uniform primary partition (1.14), we can set $R=1$ in (2.14), ..., (2.19) to deduce from Lemma 2.2 that, for this case, the estimates (2.12), (2.13) hold, with

$$\begin{aligned} C_{1,\,1} &\approx 0.38490, & C_{1,\,2} &\approx 0.09623, \\ C_{2,\,1} &\approx 0.30984, & C_{2,\,2} &\approx 0.15492, \\ C_{3,\,1} &\approx 0.18590, & C_{3,\,2} &\approx 0.25811. \end{aligned} \tag{2.46}$$

(b) Suppose that, in addition to the choice of the uniform primary partition (1.14), the secondary knots $\{x_{2i+1} : i=0, 1, ..., n-1\}$ are chosen to satisfy

$$x_{2i+1} = \frac{\xi_{i+1}+\xi_i}{2}, \qquad i=0, 1, ..., n-1. \tag{2.47}$$

Then the interval lengths in the second line of (2.20) reduce to $\alpha=\beta=\gamma=H$, $h=H/2$, which, substituted into (2.23), gives the values

$$A=0, \qquad B=\frac{1}{2H}, \qquad C=\frac{7}{4H^2}, \qquad E=\frac{3}{4H^2}, \qquad F=\frac{1}{4H^2}. \tag{2.48}$$

If we now use the values (2.48) in (2.22), we can calculate from (2.24), (2.27), (2.30), (2.36), (2.39) and (2.42) that, for the uniform partition (1.14), (2.47), the estimates (2.12), (2.13) hold, with the (mostly significantly smaller than those in (2.46)) error constants

$$\begin{aligned} C_{1,\,1} &\approx 0.29096, & C_{1,\,2} &\approx 0.09375 \\ C_{2,\,1} &\approx 0.15692, & C_{2,\,2} &\approx 0.14062, \\ C_{3,\,1} &\approx 0.08030, & C_{3,\,2} &\approx 0.07031. \end{aligned} \tag{2.49}$$

(c) Observe from (2.14), ..., (2.19) that the constants $C_{1,\,1}$, $C_{1,\,2}$, ..., $C_{3,\,2}$ are all of order $O(R)$ for $R=R_n \to \infty$, $n \to \infty$.

3. THE ENDPOINT INTERVALS

Next we bound the error $f - Vf$ in the endpoint intervals $[a, \xi_1]$ and $[\xi_{n-1}, b]$, where, from (1.3), (1.4), (1.5), it is clear that Vf coincides with Lagrange polynomial interpolant, i.e.,

$$(Vf)(x) = \begin{cases} (Lf)(x) := \sum_{i=0}^{2} \ell_i(x)\, f(\xi_i), & a \leqslant x \leqslant \xi_1, \\ (\hat{L}f)(x) := \sum_{i=n-2}^{n} \hat{\ell}_i(x)\, f(\xi_i), & \xi_{n-1} \leqslant x \leqslant b, \end{cases} \tag{3.1}$$

with the Lagrange fundamental polynomials ℓ_i and $\hat{\ell}_i$ given by

$$\begin{aligned} \ell_i(x) &= \prod_{i \neq k=0}^{2} \frac{x - \xi_k}{\xi_i - \xi_k}, && i = 0, 1, 2, \\ \hat{\ell}_i(x) &= \prod_{n-i \neq k=0}^{2} \frac{x - \xi_{n-k}}{\xi_i - \xi_{n-k}}, && i = n-2, n-1, n. \end{aligned} \tag{3.2}$$

The following estimates then hold for the endpoint intervals.

LEMMA 3.1. *Let $r \in \{1, 2, 3\}$ and suppose $f \in C^r[a, b]$. Then*

$$\max_{\substack{a \leqslant x \leqslant \xi_1 \\ \xi_{n-1} \leqslant x \leqslant b}} |(f - Vf)(x)| \leqslant d_r H^r \, \|f^{(r)}\|_\infty, \qquad r = 1, 2, 3, \tag{3.3}$$

where the positive numbers $d_r = d_r(R)$ are given by

$$\begin{aligned} d_1 &= \frac{2[(R-1)(2R+1)(R+2) + 2(R^2+R+1)^{3/2}]}{27R^2}, \\ d_2 &= \frac{27R^4 + 72R^3 + 56R^2 - 32R - 16 + (R+2)(9R^2+4R+4)^{3/2}}{1024R^2(R+1)}, \\ d_3 &= \frac{1}{9\sqrt{3}}. \end{aligned} \tag{3.4}$$

Proof. We merely give a sketch of the proof by virtue of its similarity to the (often more complicated) proofs of Lemmas 2.1 and 2.2. Also, we prove (3.3) only for $x \in [a, \xi_1]$, since the proof for $x \in [\xi_{n-1}, b]$ proceeds in an analogous manner.

For a fixed $x \in [a, \xi_1]$, it is clear from (3.1) that the sum in the right hand side of (1.17) is replaced by the sum $\sum_{i=0}^{2} \ell_i(x)(\xi_i - \theta)_+^{r-1}$. It can then be shown, using the definition (1.15), that

$$\int_a^b |K_r(x, \theta)| \, d\theta$$
$$\leqslant \frac{1}{r!} [\ell_0(x)(x-a)^r + (-1)^r \ell_1(x)(x-\xi_1)^r - (-1)^r \ell_2(x)(x-\xi_2)^r], \tag{3.5}$$

after having made use also of the sign properties, as implied by (3.2), of the Lagrange fundamental polynomials ℓ_i.

Suppose $r \in \{1, 2\}$, and introduce the notation (2.20) with $j = 0$, whence

$$y := x - a, \qquad \beta := \xi_1 - a, \qquad \gamma := \xi_2 - \xi_1. \tag{3.6}$$

Then, using (3.5), (3.2), together with the facts that $\sum_{i=0}^{2} (x - \xi_i) \ell_i(x) = 0$ and $\sum_{i=0}^{2} (x - \xi_i)^2 \ell_i(x) = 0$, we can deduce the estimates

$$\int_a^b |K_r(x, \theta)| \, d\theta \leqslant \begin{cases} \max\limits_{0 \leqslant y \leqslant \beta} p(y), & r = 1, \\ \max\limits_{0 \leqslant y \leqslant \beta} q(y), & r = 2, \end{cases} \tag{3.7}$$

where

$$p(y) = \frac{2y(\beta - y)(\beta + \gamma - y)}{\beta\gamma}, \qquad q(y) = \frac{y(\beta - y)(\beta + \gamma - y)^2}{\gamma(\beta + \gamma)}.$$

Here

$$\max_{0 \leqslant y \leqslant \beta} p(y) = p\left(\frac{2\beta + \gamma - \sqrt{\beta^2 + \beta\gamma + \gamma^2}}{3}\right) = \frac{2}{27} g(\beta, \gamma), \tag{3.8}$$

where

$$g(\beta, \gamma) = \frac{(\beta - \gamma)(2\beta + \gamma)(\beta + 2\gamma) + 2(\beta^2 + \beta\gamma + \gamma^2)^{3/2}}{\beta\gamma}, \tag{3.9}$$

and for which the bound

$$g(\beta, \gamma) \leqslant g\left(\beta, \frac{\beta}{R}\right) \leqslant R\left[\left(1 - \frac{1}{R}\right)\left(2 + \frac{1}{R}\right)\left(1 + \frac{2}{R}\right) + 2\left(1 + \frac{1}{R} + \frac{1}{R^2}\right)^{3/2}\right] H \tag{3.10}$$

can be deduced. Also,

$$\max_{0 \leqslant y \leqslant \beta} q(y) = q\left(\frac{5\beta + 2\gamma - \sqrt{9\beta^2 + 4\beta\gamma + 4\gamma^2}}{8}\right)$$

$$= \frac{\begin{bmatrix} 27\beta^4 + 72\beta^3\gamma + 56\beta^2\gamma^2 \\ \qquad - 32\beta\gamma^3 - 16\gamma^4 + (\beta + 2\gamma)(9\beta^2 + 4\beta\gamma + 4\gamma^2)^{3/2} \end{bmatrix}}{1024\gamma(\beta + \gamma)}$$

$$\leqslant \frac{R^2}{1024(R+1)}\left[27 + \frac{72}{R} + \frac{56}{R^2} - \frac{32}{R^3} - \frac{16}{R^4} \right.$$

$$\left. + \left(1 + \frac{2}{R}\right)\left(9 + \frac{4}{R} + \frac{4}{R^2}\right)^{3/2}\right] H^2. \tag{3.11}$$

The desired result (3.3) for $r = 1, 2$ is now obtained by combining (3.7), (3.8), (3.10), (3.11), and the Peano kernel estimate (1.16).

Finally, for the case $r = 3$, i.e., $f \in C^3[a, b]$, we appeal directly to a standard error estimate for quadratic Lagrange polynomial interpolation (see, e.g., [12, Theorem 1, p. 52]), according to which, in the notation (3.6), we have

$$|(f - Vf)(x)| = |(f - Lf)(x)| \leqslant \frac{\beta\gamma}{12}\, p(y)\, \|f'''\|_\infty, \qquad 0 \leqslant y \leqslant \beta,$$

with the polynomial $p(y)$ defined as in (3.7). Hence, using (3.8), (3.9) we find that

$$\max_{a \leqslant x \leqslant \xi_1} |(f - Vf)(x)| \leqslant \frac{1}{162}\, k(\beta, \gamma)\, \|f'''\|_\infty, \tag{3.12}$$

where

$$k(\beta, \gamma) = (\beta - \gamma)(2\beta + \gamma)(\beta + 2\gamma) + 2(\beta^2 + \beta\gamma + \gamma^2)^{3/2}. \tag{3.13}$$

But, since $(\partial/\partial\gamma)\, k(\beta, \gamma) > 0$ for $\beta > 0$, $\gamma > 0$, we get

$$\max_{0 \leqslant \gamma \leqslant \beta} k(\beta, \gamma) = k(\beta, \beta) = 6\sqrt{3}\,\beta^3, \qquad \beta > 0, \tag{3.14}$$

whereas clearly, also from (3.13),

$$k(\beta, \gamma) \leqslant 6\sqrt{3}\,\gamma^3 \qquad \text{if} \quad \gamma \geqslant \beta. \tag{3.15}$$

The estimate (3.3), (3.4) for $r = 3$ now follows by combining the inequalities $\beta^3 \leqslant H^3$, $\gamma^3 \leqslant H^3$, with (3.13), (3.14), (3.15), and (3.12). ∎

Remarks. (a) For the case of the uniform primary partition (1.14), we can set $R=1$ in (3.4) to calculate the error constants

$$d_1 \approx 0.76980, \qquad d_2 \approx 0.15492, \qquad d_3 \approx 0.06415. \tag{3.16}$$

(b) Observe from (3.4) that the positive number d_3 is independent of R, whereas, if $r \in \{1, 2\}$, $d_r = O(R)$ for $R = R_n \to \infty$, $n \to \infty$.

(c) It is clear, from the arguments leading to (3.10) and (3.11), as well as (3.14) and (3.15), that the estimates (3.3), (3.4) are sharp in the uniform primary partition case (1.14), for which $R=1$, but are not sharp for non-uniform primary partitions (1.2) of the interval $[a, b]$.

4. THE CASE $f(x) = x^3$

We proceed to bound the interpolation error $f - Vf$ in the case where the approximation operator V is applied to the cubic polynomial $f(x) = x^3$, $a \leqslant x \leqslant b$, i.e., f is the monomial of lowest degree which is not reproduced by V, not merely for the independent interest of such a result, but also since this result will later be needed, as is already suggested by the case $r=3$ in Lemma 2.1, for the proof of our main result in Theorem 5.1.

THEOREM 4.1. *For the monomial $f(x) = x^3$, $a \leqslant x \leqslant b$, the quadratic nodal spline interpolation error $f - Vf$ satisfies the estimate*

$$\|f - Vf\|_\infty \leqslant \frac{2}{3\sqrt{3}} H^3. \tag{4.1}$$

Proof. First, fix the index $j \in \{1, 2, ..., n-2\}$, and suppose that $x \in [\xi_j, \xi_{j+1}]$. Then, with the notation $e(x) := (f - Vf)(x)$, $a \leqslant x \leqslant b$, it is clear that $e(x)$ is a cubic polynomial with leading coefficient $=1$ on each of the intervals $[\xi_j, x_{2j+1}]$ and $[x_{2j+1}, \xi_{j+1}]$, and since $e(\xi_j) = 0 = e(\xi_{j+1})$ by virtue of the interpolation property (1.10) of V, it is easily seen (from the Taylor expansions of the polynomial pieces) that $e(x)$ can be represented on $[\xi_j, \xi_{j+1}]$ by the expressions

$$e(x) = \begin{cases} (x-\xi_j)[A + B(x-\xi_j) + (x-\xi_j)^2], & \xi_j \leqslant x \leqslant x_{2j+1}, \\ (x-\xi_{j+1})[C + D(x-\xi_{j+1}) + (x-\xi_{j+1})^2], & x_{2j+1} \leqslant x \leqslant \xi_{j+1}, \end{cases} \tag{4.2}$$

where

$$A = e'(\xi_j), \qquad B = \tfrac{1}{2}e''(\xi_j^+), \qquad C = e'(\xi_{j+1}), \qquad D = \tfrac{1}{2}e''(\xi_{j+1}^-). \tag{4.3}$$

Here it should be observed that the symbols A, B, C in (4.2), (4.3) have different definitions from those in (2.22), (2.23), as previously used in the proof of Lemma 2.2.

Also, by (1.3), (1.4), and (1.5) we have, for $\xi_j \leqslant x \leqslant \xi_{j+1}$, that

$$e(x) = x^3 - \sum_{i=j-1}^{j+2} s_i(x)\, \xi_i^3, \tag{4.4}$$

from which, using (2.2), we get

$$e(x) = \begin{cases} x^3 - \xi_{j-1}^3 + \sum\limits_{i=j}^{j+2} s_i(x)[\xi_{j-1}^3 - \xi_i^3], \\ x^3 - \xi_{j+2}^3 + \sum\limits_{i=j-1}^{j+1} s_i(x)[\xi_{j+2}^3 - \xi_i^3]. \end{cases} \tag{4.5}$$

Employing the notation (2.20), we now insert the formulas (1.6), (1.7), (1.8) into (4.5) to find that (4.2), (4.3) can be rewritten in the form

$$e(x) = \begin{cases} p(y), & 0 \leqslant y \leqslant h, \\ q(z), & -(\beta - h) \leqslant z \leqslant 0, \end{cases} \tag{4.6}$$

with

$$\begin{aligned} p(y) &= Ay + By^2 + y^3, \\ q(z) &= Cz + Dz^2 + z^3, \end{aligned} \tag{4.7}$$

and where

$$\begin{aligned} A &= -\alpha\beta, \qquad B = \alpha - \beta + \frac{1}{2}\left(\frac{\beta}{h} - 1\right)(\alpha + \beta + \gamma), \\ C &= -\beta\gamma, \qquad D = \beta - \gamma + \frac{1}{2}\left(\frac{1}{1 - \beta/h}\right)(\alpha + \beta + \gamma). \end{aligned} \tag{4.8}$$

Standard calculus techniques can now be employed to show that $p(y)$ has a local maximum at $y = y_-$, with $p(y_-) > 0$, and a local minimum at $y = y_+$, with $p(y_+) < 0$, and similarly, that $q(z)$ has a local maximum at $z = z_-$, with $q(z_-) > 0$ and a local minimum at $z = z_+$, with $q(z_+) < 0$, where $y_\pm = \frac{1}{3}(-B \pm \sqrt{B^2 - 3A})$ and $z_\pm = \frac{1}{3}(-D \pm \sqrt{D^2 - 3C})$. But, since A and C are both negative from (4.8), we deduce that $y_- < 0 < y_+$, $z_- < 0 < z_+$, and thus, since also $e \in C^1[\xi_j, \xi_{j+1}]$, with $e(\xi_j) = 0 = e(\xi_{j+1})$, we can now use (4.6) to argue that the inequalities $y_+ \leqslant h$ and $z_- \geqslant -(b - h)$ must hold. In particular, if $y_+ > h$, then clearly, since

$p(y_-) > 0$ and $p(0) = 0$, we must have $p(h) < 0$ and $p'(h) < 0$. But, by virtue of (4.6) and the fact that e is a continuously differentiable function on $[\xi_j, \xi_{j+1}]$, we have that $q(-(\beta - h)) = p(h)$ and $q'(-(\beta - h)) = p'(h)$, so that then also $q(-(\beta - h)) < 0$ and $q'(-(\beta - h)) < 0$, which in turn imply, together with the fact that $q(0) = 0$, that $q(z)$ has a local minimum in the interval $(-(\beta - h), 0)$, thereby contradicting the known fact that $q(z)$ has a unique local minimum at $z_+ > 0$. The possibility $z_- < -(\beta - h)$ is similarly eliminated. Hence

$$\max_{\xi_j \leqslant x \leqslant \xi_{j+1}} |e(x)| = \max\{-p(y_+), q(z_-)\}. \tag{4.9}$$

But, from (4.7), we get

$$\begin{aligned} -p(y_+) &= \tfrac{1}{27}[9AB - 2B^3 + 2(B^2 - 3A)^{3/2}], \\ q(z_-) &= \tfrac{1}{27}[2D^3 - 9DC + 2(D^2 - 3C)^{3/2}]. \end{aligned} \tag{4.10}$$

Recalling from (4.8) that $C < 0$, it is easily seen that the right-hand side of the second equation in (4.10) is, for a given C, a strictly *increasing* function of D for all real D, whereas a differentiation procedure shows that the right-hand side of the first equation in (4.10) is, for a given A, a strictly *decreasing* function of B for all real B. Also, from (4.8), we have the bounds

$$\left.\begin{aligned} B &> \alpha - \beta \\ D &< \beta - \gamma \end{aligned}\right\}, \qquad 0 < h < \beta, \tag{4.11}$$

so that, inserting (4.11), and the values in (4.8) for A and C into (4.10), we obtain the bounds

$$\begin{aligned} -p(y_+) &< \tfrac{1}{27}[(\beta - \alpha)(2\alpha + \beta)(\alpha + 2\beta) + 2(\alpha^2 + \alpha\beta + \beta^2)^{3/2}, \\ q(z_-) &< \tfrac{1}{27}[(\beta - \gamma)(2\beta + \gamma)(\beta + 2\gamma) + 2(\beta^2 + \beta\gamma + \gamma^2)^{3/2}]. \end{aligned} \tag{4.12}$$

Employing a technique similar to the one following (3.12) and leading to (3.15), we can then deduce from (4.12) and (4.9) that

$$\max_{\xi_j \leqslant x \leqslant \xi_{j+1}} |e(x)| \leqslant \frac{2}{3\sqrt{3}} H^3, \qquad j = 1, 2, ..., n-2. \tag{4.13}$$

Next, we choose $f(x) = x^3$, $a \leqslant x \leqslant b$, in Lemma 3.1, for which, in the context of that result, we have $r = 3$, so that (3.3) and (3.4) yield the estimate

$$\max_{\substack{a \leqslant x \leqslant \xi_1 \\ \xi_{n-1} \leqslant x \leqslant b}} |e(x)| \leqslant \frac{2}{3\sqrt{3}} H^3. \tag{4.14}$$

Finally, we combine (4.13) and (4.14) to obtain the desired result (4.1). ▮

Remarks. (a) Note that, by virtue of the fact that the inequalities in (4.11) are strict, the bound (4.1) is not sharp.

(b) For the uniform primary partition (1.14), together with the choice (2.47) for the secondary knots $\{x_{2i+1} : i=0, 1, ..., n-1\}$, the constants A, B, C and D in (4.8) can be shown from (2.20) to be given by

$$A=-H^2, \qquad B=\tfrac{3}{2}H, \qquad C=-H^2, \qquad D=-\tfrac{3}{2}H. \tag{4.15}$$

Hence, substituting (4.15) into (4.10), and then using (4.9), we find, for $f(x)=x^3$, $a \leqslant x \leqslant b$, that

$$\max_{\xi_j \leqslant x \leqslant \xi_{j+1}} |(f-Vf)(x)| = \left[-\frac{3}{4}+\frac{7}{12}\sqrt{\frac{7}{3}}\right] H^3 \approx 0.14106 H^3,$$
$$j=1, 2, ..., n-2, \tag{4.16}$$

which clearly yields a significantly smaller bound on the interior interval $[\xi_1, \xi_{n-1}]$ than the maximum norm bound $2/(3\sqrt{3})\, H^3 \approx 0.38490 H^3$ in (4.1).

(c) Furthermore, observe that the bound in (4.1) is independent of R, so that, in particular, for the case $f(x)=x^3$, $a \leqslant x \leqslant b$, we have $\|f-Vf\|_\infty = O(H^3) \to 0$ if $H=H_n \to 0$, $n \to \infty$, even if $R=R_n \to \infty$, $n \to \infty$.

5. JACKSON-TYPE ESTIMATES FOR $\|f-Vf\|_\infty$

Finally, we combine the results of Lemmas 2.1, 2.2, 3.1, and Theorem 4.1 to derive Jackson-type bounds for quadratic nodal spline interpolation.

To bound $|(f-Vf)(x)|$ for a given $x \in [\xi_j, \xi_{j+1}]$ with $j \in \{0, 1, ..., n-1\}$, we appeal to Lemma 3.1 if $j=0$ or $j=n-1$, whereas, for $j \in \{1, ..., n-2\}$, we recall the results of Lemmas 2.1 and 2.2, in particular using (2.3) for $x \in [\xi_j, x_{2j+1}]$ and (2.4) for $x \in [x_{2j+1}, \xi_{j+1}]$, as well as the estimate (4.1) from Theorem 4.1 for the case $r=3$ in (2.3), (2.4).

Since the error constants appearing in Lemmas 2.2 and 3.1 can furthermore be shown to satisfy, for $R \geqslant 1$, the inequalities

$$d_1 < 2(C_{1,1}+C_{1,2}), \qquad d_2 < C_{2,1}+C_{2,2},$$

we have now proved our main result which can be formulated as follows.

THEOREM 5.1. *Suppose $f \in C^r[a, b]$ with $r \in \{1, 2, 3\}$. Then the quadratic nodal spline interpolation error $f - Vf$ satisfies the estimates*

$$\|f - Vf\|_\infty \leqslant M_r H^r \|f^{(r)}\|_\infty, \tag{5.1}$$

where the positive numbers $M_r = M_r(R)$ are given by

$$\begin{aligned} M_1 = 2(C_{1,1} + C_{1,2}), \qquad M_2 = C_{2,1} + C_{2,2}, \\ M_3 = \frac{1}{9\sqrt{3}} + \frac{1}{3}(C_{3,1} + C_{3,2}), \end{aligned} \tag{5.2}$$

in terms of the positive numbers $C_{1,1}, ..., C_{3,2}$ as given in Lemma 2.2.

Remarks. (a) For the case of the uniform primary partition (1.14), we see from (5.2) and (2.46) that the estimates (5.1) hold with

$$M_1 \approx 0.96225, \qquad M_2 \approx 0.46476, \qquad M_3 \approx 0.21216. \tag{5.3}$$

(b) If, in addition to the uniformity condition (1.14) for the primary knots $\{\xi_i : i = 0, 1, ..., n\}$, the secondary knots $\{x_{2i+1} : i = 0, 1, ..., n-1\}$ are chosen to satisfy (2.47), we can insert the values (2.49) into (5.2), and replace the constant $1/(9\sqrt{3})$ in (5.2) by the smaller value $(-1/8 + (7/72)\sqrt{7/3})$ implied by (2.3) and (4.16), to deduce that, in this case, the estimates (5.1) hold with (the significantly smaller than those in (5.3)) error constants

$$M_1 \approx 0.77436, \qquad M_2 \approx 0.29754, \qquad M_3 \approx 0.07371. \tag{5.4}$$

(c) Note that (cf. Remark (c) after Lemma 4.2) the error constants $M_r = O(R)$ for $R = R_n \to \infty, n \to \infty$.

(d) Observe that the estimates (5.1) are independent of the choice of the secondary knots $\{x_{2i+1} : i = 0, 1, ..., n\}$.

Conclusion. Finally, we can now compare our results (5.1), (5.2), (5.3), (5.4) with those previously obtained in [7, 9], where it was established that the estimates (5.1) hold with $M_r = M_r(R)$ replaced by $k_r = k_r(R)$, as given by

$$\begin{aligned} k_1 = \left[2 + \frac{R^2}{2(1+R)}\right]\frac{\pi}{4}, \qquad k_2 = \left[2 + \frac{R^2}{2(1+R)}\right]\frac{3\pi^2}{32}, \\ k_3 = \left[2 + \frac{R^2}{2(1+R)}\right]\frac{9}{64}, \end{aligned} \tag{5.5}$$

yielding, for the uniform primary partition case (1.14), so that $R=1$ in (5.5), the values

$$k_1 \approx 1.76715, \qquad k_2 \approx 2.08187, \qquad k_3 \approx 0.31641. \tag{5.6}$$

Hence, for the case of the uniform primary partition (1.14), and even more so if (1.14), (2.47) are both satisfied, we see from (5.3), (5.4) and (5.6) that our error constants M_r are significantly smaller than the previously obtained error constants k_r. Moreover, it can be shown that

$$M_r(R) < k_r(R) \qquad \text{for} \quad \begin{cases} R \geqslant 1 & \text{if} \quad r=2,3, \\ 1 \leqslant R \leqslant 4.04733 & \text{if} \quad r=1, \end{cases}$$

which, together with the above mentioned significant improvement obtained for uniform partitions, clearly illustrates the usefulness of Peano kernel techniques for interpolation error analysis as opposed to the cruder estimation methods based on estimating the Lebesgue constant $\|V\|_\infty$, as employed in [7, 9].

ACKNOWLEDGMENT

The authors are most grateful to Carl Rohwer for his inspiration on the topic of nodal splines and for many fruitful discussions during the preparation of this paper.

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